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Higher-order correction to the Casimir force on a compact ball when $\varepsilon\mu = 1$

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Abstract. The Casimir surface force density on a compact spherical ball composed of matter satisfying the condition $\varepsilon\mu = 1$ (ε being the permittivity and μ the permeability) is calculated. The matter is taken to be non-dispersive. Schwinger's source theory is employed. The present calculation improves a previous calculation of the same effect by Brevik and Kolbenstvedt by four orders of magnitude in the Debye expansion. As before, the surface force is found to be repulsive.

1. Introduction

The Casimir effect is the change in the electromagnetic vacuum field experienced when the infinite 'ground state' of the vacuum becomes disturbed by some kind of external constraint. Usually, one idealises the situation in the sense that the constraint is taken to be an ideal boundary, such as the surface of a metal of infinite conductivity. A characteristic feature of the development of this field of research in recent years is the increasing emphasis that has been laid on the need to also take the properties of real *material media* into consideration. A real medium is endowed with permittivity and permeability as well as dispersive properties at high frequencies. According to the Kramers–Kronig formulae (Landau and Lifshitz 1984) a material concept such as the permittivity must in general even be complex.

Progress in this field of research is usually made by adopting one or more simplifying conditions as regards the properties of the medium. In the present paper we shall examine the Casimir effect under the following set of assumptions. There is an isotropic and homogeneous medium present, in the form of a compact spherical ball of radius a. The material is assumed *non-dispersive*, it is at zero absolute temperature and it satisfies the condition

$$\varepsilon\mu = 1 \tag{1.1}$$

where the permittivity ε and the permeability μ are real quantities. This particular condition has quite remarkable properties. Mathematically, it eliminates cutoff problems that otherwise plague Casimir theories in the presence of ordinary dielectric media (Milton 1980, Brevik 1982a). Physically, the condition is exactly the one required in Lee's theory of the vacuum (Lee 1981) to ensure the gluon velocity to be equal to the velocity of light. There are accordingly both mathematical and physical reasons for studying media satisfying the condition (1.1). We have earlier studied various aspects of this kind of theory (Brevik and Kolbenstvedt 1982a, b, 1983, 1984, 1985, Brevik 1982b). In one of these references (Brevik and Kolbenstvedt 1982b, hereafter referred to as BK) we derived the following formula for the surface force density on the sphere:

$$F = F_0\left(\frac{\mu-1}{\mu+1}\right)^2\left(1+0.311\frac{\mu}{(\mu+1)^2}\right). \tag{1.2}$$

Here

$$F_0 = 0.092\,35/8\pi a^4 \tag{1.3}$$

is the result derived by Milton *et al* (1978) for a perfectly conducting shell in vacuum, and

$$\mu = \mu_1/\mu_2 \tag{1.4}$$

where μ_1 is the inside permeability and μ_2 the outside permeability. The purpose of the present paper is to improve on the accuracy of the result (1.2) by carrying out the Debye expansion of the Riccati–Bessel functions four orders of magnitude further. It turns out that the resulting formula for the force exhibits the same essential properties as were found previously in BK: the force is repulsive; it is moreover invariant under the interchange between the inside and the outside medium and it approaches the value F_0 if $\mu \to 0$ or $\mu \to \infty$. We shall also have the opportunity to demonstate by an example how the region of applicability of the Debye expansion, which after all is an asymptotic expansion, is limited. In the appendix we have collected useful information about the high-order polynomials occurring in the Debye expansion.

Readers interested in general review papers on the Casimir effect are referred to the recent works of Plunien *et al* (1986) and DeRaad (1985), the latter dealing with Schwinger's source theory.

In this paper, $\hbar$ and c are put equal to unity.

2. Calculation by means of the Debye expansion

The general expression for the surface force density on the sphere, cf BK, can be written as follows:

$$F = -\frac{1}{4\pi^2 a^4}\int_0^\infty \mathrm{d}x\,\cos(\delta x)x\sum_{l=1}^{\infty}\frac{\nu(\mathrm{d}/\mathrm{d}x)\ln(1-\lambda_l^2(x))}{(1-\chi^{-1}/s_l(x)e_l'(x))(1+\chi^{-1}/s_l'(x)e_l(x))} \tag{2.1}$$

with $\nu = l+\frac{1}{2}$. This expression follows from application of the Maxwell stress tensor and the electromagnetic boundary conditions at $r=a$; there is thus no mathematical approximation involved at this stage. In the derivation of (2.1) we have performed a complex frequency rotation:

$$k \to \mathrm{i}\hat{k} = \mathrm{i}\hat{\omega} \qquad \tau \to \mathrm{i}\hat{\tau} \tag{2.2}$$

with $\tau = t - t'$ denoting the temporal splitting of the two spacetime points involved. Further, δ is the non-dimensional cutoff parameter and x the non-dimensional frequency; they are defined as

$$\delta = \hat{\tau}/a \qquad x = \hat{k}a \tag{2.3}$$

with $x \geqslant 0$. The remaining quantities in (2.1) are defined as

$$\chi = \mu - 1 \tag{2.4}$$

$$s_l(x) = (\pi x/2)^{1/2} I_\nu(x) \tag{2.5a}$$

$$e_l(x) = (2x/\pi)^{1/2} K_\nu(x) \tag{2.5b}$$

$$\lambda_l(x) = [s_l(x) e_l(x)]'. \tag{2.6}$$

The Wronskian of the Riccati–Bessel functions defined according to (2.5) is $W\{s_l, e_l\} = -1$.

We now calculate (2.1) analytically using the Debye expansion. Our first task is to expand the inverse of the denominator to the order ν^{-6}. The Debye expansion can in general be written as follows (Abramowitz and Stegun 1970):

$$s_l(x) = \frac{1}{2}\frac{z^{1/2}}{(1+z^2)^{1/4}} \exp(\nu\eta)\left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}\right) \tag{2.7}$$

$$e_l(x) = \frac{z^{1/2}}{(1+z^2)^{1/4}} \exp(-\nu\eta)\left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k}\right) \tag{2.8}$$

$$s_l'(x) = \frac{1}{2}\frac{(1+z^2)^{1/4}}{z^{1/2}} \exp(\nu\eta)\left(1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k}\right) \tag{2.9}$$

$$e_l'(x) = -\frac{(1+z^2)^{1/4}}{z^{1/2}} \exp(-\nu\eta)\left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{\nu^k}\right) \tag{2.10}$$

where we have defined

$$z = x/\nu \qquad t(z) = (1+z^2)^{-1/2}. \tag{2.11}$$

The values of u_k and v_k up to $k = 6$ are given as polynomials in t, with exact fractional coefficients, in the appendix. The value of η will not be needed in the following.

We now expand the inverse of the denominator in (2.1), making use of the relatively simple expressions for $s_l e_l'$ and $s_l' e_l$ in the appendix. A lengthy calculation gives

$$\begin{aligned}
&\frac{1}{(1-\chi^{-1}/s_l e_l')(1+\chi^{-1}/s_l' e_l)} \\
&\quad = \left(\frac{\mu-1}{\mu+1}\right)^2 \left\{1 - \frac{t^6}{\nu^2}\frac{\mu}{(\mu+1)^2}\right. \\
&\qquad + \frac{t^6}{\nu^4}\frac{\mu}{4(\mu+1)^2}\left[2 - 27t^2 + 60t^4 - 35t^6 - \left(\frac{\mu-1}{\mu+1}\right)^2 t^6\right] \\
&\qquad - \frac{t^6}{\nu^6}\frac{\mu}{16(\mu+1)^2}\left(1 - 54t^2 + 1146t^4 - 6204t^6 + 13\,239t^8 - 12\,258t^{10}\right. \\
&\qquad \left. + 4131t^{12} + \frac{8\mu t^6}{(\mu+1)^2}(2 - 27t^2 + 60t^4 - 35t^6) - \frac{8\mu(\mu^2+1)}{(\mu+1)^4}t^{12}\right) \\
&\qquad \left. + \mathrm{O}(1/\nu^8)\right\}.
\end{aligned} \tag{2.12}$$

We define the following integrals:

$$J(l,\delta)=\frac{2\nu^2}{\pi}\int_0^\infty \mathrm{d}z\cos(\delta\nu z)z\frac{\mathrm{d}}{\mathrm{d}z}\ln(1-\lambda_l^2) \tag{2.13}$$

$$I(l,\delta)=\frac{2\nu^2}{\pi}\int_0^\infty \mathrm{d}z\cos(\delta\nu z)zt^6\frac{\mathrm{d}}{\mathrm{d}z}\ln(1-\lambda_l^2) \tag{2.14}$$

$$K(l,\delta)=\frac{\nu^2}{2\pi}\int_0^\infty \mathrm{d}z\cos(\delta\nu z)zt^6\left[2-27t^2+60t^4-35t^6-\left(\frac{\mu-1}{\mu+1}\right)^2t^6\right]\frac{\mathrm{d}}{\mathrm{d}z}\ln(1-\lambda_l^2) \tag{2.15}$$

$$L(l,\delta)=\frac{\nu^2}{8\pi}\int_0^\infty \mathrm{d}z\cos(\delta\nu z)zt^6\bigg(1-54t^2+1146t^4-6204t^6+13\,239t^8 -12\,258t^{10}+4131t^{12}+\frac{8\mu t^6}{(\mu+1)^2}(2-27t^2+60t^4-35t^6) -\frac{8\mu(\mu^2+1)}{(\mu+1)^4}t^{12}\bigg)\frac{\mathrm{d}}{\mathrm{d}z}\ln(1-\lambda_l^2) \tag{2.16}$$

so that the expression (2.1) can be written as follows:

$$F=-\frac{1}{8\pi a^4}\left(\frac{\mu-1}{\mu+1}\right)^2\sum_{l=1}^\infty\bigg(J(l,\delta)-\frac{1}{\nu^2}\frac{\mu}{(\mu+1)^2}I(l,\delta)+\frac{1}{\nu^4}\frac{\mu}{(\mu+1)^2}K(l,\delta) -\frac{1}{\nu^6}\frac{\mu}{(\mu+1)^2}L(l,\delta)+\mathrm{O}(1/\nu^8)\bigg). \tag{2.17}$$

The two first of these integrals, $J(l,\delta)$ and $I(l,\delta)$, are defined such as in BK.

We shall not calculate the sum of $J(l,\delta)$ here since this term, which is the only term occurring in the case of a conducting shell in vacuum, was calculated accurately by Milton *et al* (1978). These authors employed numerical methods for the lowest values of l and the Debye expansion thereafter, obtaining

$$\sum_{l=1}^\infty J(l,\delta)=-0.092\,35. \tag{2.18}$$

In this calculation, the cutoff parameter δ played an important role. We shall adopt the value (2.18) in the following.

Our remaining task is to calculate the sum over l of the three integrals relating to the material medium, namely $I(l,\delta)$, $K(l,\delta)$ and $L(l,\delta)$. It turns out that in these cases the parameter δ is without significance. We thus put $\delta=0$ in the following. We intend to calculate these integrals to an accuracy of $\mathrm{O}(1/\nu^6)$. From (A11) it is apparent that the leading term in λ_l is of $\mathrm{O}(1/\nu)$. An expansion in λ_l as the smallness parameter is thus equivalent to an expansion in $1/\nu$, which is in accordance with the spirit of the Debye expansion. First expanding the logarithm:

$$\begin{aligned}\ln(1-\lambda_l^2)&=-\lambda_l^2-\tfrac{1}{2}\lambda_l^4-\tfrac{1}{3}\lambda_l^6+\mathrm{O}(\lambda_l^8)\\ &=-\frac{t^6}{4\nu^2}\bigg(1-\frac{1}{8\nu^2}(4-54t^2+120t^4-71t^6)+\frac{1}{48\nu^4}(3-162t^2+3438t^4\\ &\quad-18\,606t^6+39\,636t^8-36\,594t^{10}+12\,286t^{12})+\mathrm{O}(1/\nu^6)\bigg)\end{aligned} \tag{2.19}$$

and thereafter differentiating the expression with respect to z, observing that $\mathrm{d}t/\mathrm{d}z = -zt^3$, we obtain

$$\nu^2 \frac{\mathrm{d}}{\mathrm{d}z} \ln(1-\lambda_l^2)$$

$$= \tfrac{3}{2} z t^8 \left(1 - \frac{1}{4\nu^2}(2 - 36t^2 + 100t^4 - 71t^6) \right.$$

$$\left. + \frac{1}{16\nu^4}(1 - 72t^2 + 1910t^4 - 12\,404t^6 + 30\,828t^8 - 32\,528t^{10} + 12\,286t^{12}) \right)$$

$$+ \mathrm{O}(1/\nu^6) \tag{2.20}$$

where we have applied a factor ν^2 for convenience.

We have thus found the Debye expansions that are needed to calculate $I(l,0)$, $K(l,0)$ and $L(l,0)$. Since the leading term in (2.20) is of order unity, it follows from the definition equations (2.14)-(2.16) that these integrals are also of order unity. It is convenient to split them up as follows:

$$I(l,0) = I_0 + \frac{1}{\nu^2} I_2 + \frac{1}{\nu^4} I_4 + \ldots$$

$$K(l,0) = K_0 + \frac{1}{\nu^2} K_2 + \ldots \tag{2.21}$$

$$L(l,0) = L_0 + \ldots$$

where the coefficients $I_0, I_2, \ldots,$ are independent of l. These coefficients consist essentially of beta functions. Their evaluation yields

$$I_0 = \frac{3}{\pi} \int_0^\infty \mathrm{d}z\, z^2 t^{14} = 63/2^{11} \tag{2.22}$$

$$I_2 = -\frac{3}{4\pi} \int_0^\infty \mathrm{d}z\, z^2 t^{14} (2 - 36t^2 + 100t^4 - 71t^6) = 4911/2^{18} \tag{2.23}$$

$$I_4 = \frac{3}{16\pi} \int_0^\infty \mathrm{d}z\, z^2 t^{14} (1 - 72t^2 + 1910t^4 - 12\,404t^6 + 30\,828t^8 - 32\,528\, t^{10} + 12\,286t^{12})$$

$$= -30\,675/2^{25} \tag{2.24}$$

$$K_0 = \frac{3}{4\pi} \int_0^\infty \mathrm{d}z\, z^2 t^{14} \left[2 - 27t^2 + 60t^4 - 35t^6 - \left(\frac{\mu - 1}{\mu + 1}\right)^2 t^6 \right]$$

$$= -\frac{3}{2^{18}} \left[713 + 715 \left(\frac{\mu - 1}{\mu + 1}\right)^2 \right] \tag{2.25}$$

$$K_2 = -\frac{3}{16\pi} \int_0^\infty \mathrm{d}z\, z^2 t^{14} \left[2 - 27t^2 + 60t^4 - 35t^6 - \left(\frac{\mu - 1}{\mu + 1}\right)^2 t^6 \right] (2 - 36t^2 + 100t^4 - 71t^6)$$

$$= \frac{3}{2^{26}} \left[175\,691 - 36\,439 \left(\frac{\mu - 1}{\mu + 1}\right)^2 \right] \tag{2.26}$$

$$L_0=\frac{3}{16\pi}\int_0^\infty dz\, z^2 t^{14}\bigg(1-54t^2+1146t^4-6204t^6+13\,239t^8-12\,258t^{10}$$

$$+4131t^{12}+\frac{8\mu t^6}{(\mu+1)^2}(2-27t^2+60t^4-35t^6)-\frac{8\mu(\mu^2+1)}{(\mu+1)^4}t^{12}\bigg)$$

$$=\frac{3}{2^{26}}\left(23\,475+224\,744\frac{\mu}{(\mu+1)^2}-235\,144\frac{\mu(\mu^2+1)}{(\mu+1)^4}\right). \qquad (2.27)$$

The force (2.17) can be expressed in terms of these coefficients as

$$F=\frac{1}{8\pi a^4}\left(\frac{\mu-1}{\mu+1}\right)^2\bigg(0.092\,35+\frac{\mu}{(\mu+1)^2}I_0\sum_{l=1}^{\infty}\frac{1}{\nu^2}$$

$$+\frac{\mu}{(\mu+1)^2}(I_2-K_0)\sum_{l=1}^{\infty}\frac{1}{\nu^4}$$

$$+\frac{\mu}{(\mu+1)^2}(I_4-K_2+L_0)\sum_{l=1}^{\infty}\frac{1}{\nu^6}+\sum O(1/\nu^8)\bigg). \qquad (2.28)$$

From Gradshteyn and Ryzhik (1980, p 7) we have for the first two sums

$$\sum_{l=1}^{\infty}\frac{1}{\nu^2}=\tfrac{1}{2}\pi^2-4 \qquad (2.29)$$

$$\sum_{l=1}^{\infty}\frac{1}{\nu^4}=\tfrac{1}{6}\pi^4-16 \qquad (2.30)$$

whereas the third sum can be evaluated by first rewriting it as a sum ranging from $-\infty$ to ∞ and thereafter using the calculus of residues (Morse and Feshbach 1953, p 413) to obtain

$$\sum_{l=1}^{\infty}\frac{1}{\nu^6}=\frac{1}{2}\sum_{l=-\infty}^{\infty}\frac{1}{\nu^6}-64$$

$$=-\frac{1}{2}\frac{\pi}{5!}\left(\frac{d^5}{dz^5}\cot\pi z\right)_{z=-1/2}-64$$

$$=\tfrac{1}{15}\pi^6-64. \qquad (2.31)$$

We now have at our disposal all quantities needed to evaluate (2.28). Omitting the remainder term and reintroducing F_0 from (1.3), we can conveniently express the surface force density as follows:

$$F=F_0\left(\frac{\mu-1}{\mu+1}\right)^2\bigg[1+0.372\,03\frac{\mu}{(\mu+1)^2}\left(1-0.001\,25\frac{\mu}{(\mu+1)^2}+0.056\,67\frac{\mu^2}{(\mu+1)^4}\right)$$

$$+0.022\,44\left(\frac{\mu-1}{\mu+1}\right)^2\frac{\mu}{(\mu+1)^2}\bigg] \qquad (2.32)$$

showing that the permeability occurs only in the combinations $(\mu-1)^2/(\mu+1)^2$ and $\mu/(\mu+1)^2$. We have in (2.32) chosen to give the same number of decimals as in F_0, cf (1.3). Expression (2.32) is our main result.

3. Conclusions and final remarks

(i) Our main purpose in this paper has been to calculate, by use of the Debye expansion, the surface force density on a compact spherical ball composed of matter satisfying the condition (1.1). The result, shown in (2.32), follows from an expansion of the matter-relating terms in the integrand in the general expression (2.1) to an accuracy of $O(1/\nu^6)$. Our earlier result (BK), which is reproduced in (1.2), corresponded to an accuracy of $O(1/\nu^2)$, i.e. it terminated with the I_0 term in (2.28). In the present paper there is thus an improvement of four orders in magnitude in the matter-relating terms.

(ii) The force (2.32) is generally *repulsive*. Correspondingly, the Casimir energy E is positive. Specifically, if

$$E_0 = 0.092\,35/2a \tag{3.1}$$

is the Casimir energy for a perfectly conducting shell, we have

$$E/E_0 = F/F_0. \tag{3.2}$$

With the exception of the extreme cases $\mu \to 0$ or $\mu \to \infty$, E is thus always less than E_0.

(iii) Expression (2.32) is invariant under the substitution $\mu \to 1/\mu$. Since μ is the relative inside/outside permeability, this means that the force is invariant under an interchange of the inside with the outside medium. In particular, the force on the surface of a compact ball in vacuum is the same as on the spherical boundary of a cavity, if the matter around the cavity is the same as in the ball.

(iv) It ought to be borne in mind that we have assumed the medium to be *non-dispersive*. Our theory thus falls within the same category as that of Milton *et al* (1978). The inclusion of dispersive effects in the Casimir theory of dielectric media is an interesting generalisation, which may imply important physical consequences for electrodynamics as well as for the formally similar quantum chromodynamics in the limit of vanishing quark-gluon coupling. Candelas (1982, 1986) has discussed problems of this kind; the reader may consult also the related papers by Baacke and Kasperidus (1985) and Baacke and Krüsemann (1986). Brevik and Einevoll (1987) have recently examined the consequences of a specific dispersive model.

(v) It is instructive to have a closer look at the sum $\Sigma_1^\infty J(l, \delta)$. We adopted above the result (2.18), which was calculated by Milton *et al* (1978). One may wonder what would be the result of a straightforward analytic calculation of this sum using the same method as we used above. It turns out, in fact, that such a calculation would not work beyond the second-order approximation. This can easily be seen if we make a decomposition of the term, by analogy to (2.21),

$$J(l, \delta) = J_0(\delta) + \frac{1}{\nu^2} J_2 + \frac{1}{\nu^4} J_4 + \dots \tag{3.3}$$

(it is sufficient to go to the fourth order). It is necessary to keep δ different from zero in $J_0(\delta)$ to avoid divergences. Inserting the expansion (2.20) into (2.13) we obtain for the zeroth-order term

$$J_0(\delta) = \frac{3}{\pi} \int_0^\infty \mathrm{d}z\, z^2 t^8 \cos(\delta \nu z) = \frac{3}{32}\left(3 - 3\delta \frac{\mathrm{d}}{\mathrm{d}\delta} + \delta^3 \frac{\mathrm{d}^3}{\mathrm{d}\delta^3}\right) \exp(-\delta\nu) \tag{3.4}$$

cf Gradshteyn and Ryzhik (1980, formula 3.737.5). Summing over l and afterwards letting $\delta \to 0$ we obtain

$$\lim_{\delta \to 0} \sum_{l=1}^{\infty} J_0(\delta) = -\tfrac{3}{32}. \tag{3.5}$$

Evaluating the second and third term (with $\delta=0$) in (3.3) we obtain, after summing over l,

$$J_2 \sum_{l=1}^{\infty} \frac{1}{\nu^2} = 0.001\,03 \tag{3.6}$$

$$J_4 \sum_{l=1}^{\infty} \frac{1}{\nu^4} = 4.512\,25. \tag{3.7}$$

Here the second-order result in (3.6) is reasonable: when combined with (3.5) it yields the number $-0.092\,72$ which is not very far from the number in (2.18). However, the fourth-order result in (3.7) is obviously wrong. Thus, the expansion procedure, which proved to be most useful in the evaluation of all matter-relating terms above, does not work beyond the second order as far as $J(l, \delta)$ is concerned. This term is too delicate to be calculable by the straightforward expansion method. We trace this discrepancy back to the fact that the Debye expansion is an *asymptotic* expansion; as such it is useful and accurate up to an optimum number of terms but decreases in accuracy if this optimum number is exceeded.

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Appendix. The coefficients in the Debye expansion

We need the coefficients u_k and v_k up to the sixth order. Since the high-power terms in the polynomials tend to compensate each other closely in expressions of physical interest, it is advantageous to give the coefficients in exact form, as fractional numbers. They are

$$u_1 = \frac{1}{8}t - \frac{5}{24}t^3 \qquad v_1 = \frac{1}{8}t + \frac{7}{24}t^3 \tag{A1}$$

$$u_2 = \frac{9}{128}t^2 - \frac{77}{192}t^4 + \frac{385}{1152}t^6 \tag{A2a}$$

$$v_2 = -\frac{7}{128}t^2 + \frac{79}{192}t^4 - \frac{455}{1152}t^6 \tag{A2b}$$

$$u_3 = \frac{75}{1024}t^3 - \frac{4563}{5120}t^5 + \frac{17\,017}{9216}t^7 - \frac{85\,085}{82\,944}t^9 \tag{A3a}$$

$$v_3 = -\frac{69}{1024}t^3 + \frac{13\,651}{15\,360}t^5 - \frac{18\,095}{9216}t^7 + \frac{95\,095}{82\,944}t^9 \tag{A3b}$$

$$u_4 = \frac{3675}{32\,768}t^4 - \frac{96\,833}{40\,960}t^6 + \frac{144\,001}{16\,384}t^8 - \frac{7436\,429}{663\,552}t^{10} + \frac{37\,182\,145}{7962\,624}t^{12} \tag{A4a}$$

$$v_4 = -\frac{3525}{32\,768}t^4 + \frac{96\,187}{40\,960}t^6 - \frac{6663\,371}{737\,280}t^8 + \frac{7878\,871}{663\,552}t^{10} - \frac{40\,415\,375}{7962\,624}t^{12} \tag{A4b}$$

$$u_5 = \frac{59\,535}{262\,144}t^5 - \frac{67\,608\,983}{9175\,040}t^7 + \frac{250\,881\,631}{5898\,240}t^9 - \frac{108\,313\,205}{1179\,648}t^{11} + \frac{5391\,411\,025}{63\,700\,992}t^{13} - \frac{5391\,411\,025}{191\,102\,976}t^{15} \tag{A5a}$$

$$v_5 = -\frac{58\,065}{262\,144}t^5 + \frac{67\,165\,069}{9175\,040}t^7 - \frac{254\,476\,937}{5898\,240}t^9 + \frac{1008\,167\,303}{10\,616\,832}t^{11} - \frac{5673\,995\,327}{63\,700\,992}t^{13} + \frac{5763\,232\,475}{191\,102\,976}t^{15} \tag{A5b}$$

$$u_6 = \frac{2401\,245}{4194\,304}t^6 - \frac{388\,895\,895}{14\,680\,064}t^8 + \frac{1441\,372\,804\,469}{6606\,028\,800}t^{10} - \frac{33\,010\,308\,331}{47\,185\,920}t^{12} + \frac{4445\,922\,195}{4194\,304}t^{14} - \frac{1169\,936\,192\,425}{1528\,823\,808}t^{16} + \frac{5849\,680\,962\,125}{27\,518\,828\,544}t^{18} \tag{A6a}$$

$$v_6 = -\frac{2361\,555}{4194\,304}t^6 + \frac{1933\,307\,473}{73\,400\,320}t^8 - \frac{207\,514\,649\,173}{943\,718\,400}t^{10} + \frac{6742\,901\,165}{9437\,184}t^{12} - \frac{1117\,254\,404\,695}{1019\,215\,875}t^{14} + \frac{1223\,850\,302\,675}{1528\,823\,808}t^{16} - \frac{6183\,948\,445\,675}{27\,518\,828\,544}t^{18}. \tag{A6b}$$

The values of u_k, up to $k=4$, are as given in Abramowitz and Stegun (1970). The values for higher k can be found in British Association for the Advancement of Science (1952). As regards v_k, the values up to $k=3$ follow from Abramowitz and Stegun (AS), when one takes into account that our definition of v_k is different from theirs:

$$v_k = v_k(\mathrm{AS}) + \tfrac{1}{2}tu_{k-1} \qquad k = 1, 2, \ldots . \tag{A7}$$

The reason for this difference is that our v_k refer to the *Riccati-Bessel* functions instead of to the modified Bessel functions. The v_k for higher k can be found from the recursive relations, also given by Abramowitz and Stegun. Actually, the values above were found by means of an analytic computer.

The following relations between the coefficients are useful:

$$\begin{aligned} &u_2 - u_1v_1 + v_2 = 0 \\ &u_4 - u_3v_1 + u_2v_2 - u_1v_3 + v_4 = 0 \\ &u_6 - u_5v_1 + u_4v_2 - u_3v_3 + u_2v_4 - u_1v_5 + v_6 = 0. \end{aligned} \tag{A8}$$

Whereas the Debye expansions for s_l, e_l and their derivatives are complicated, cf (2.7)-(2.10), the expansions for the combinations s_le_l' and $s_l'e_l$ are relatively simple:

$$s_le_l' = -\tfrac{1}{2}\bigg(1 - \frac{t^3}{2\nu} + \frac{1}{16\nu^3}(2t^3 - 27t^5 + 60t^7 - 35t^9) + \frac{1}{256\nu^5} \times (108t^5 - 3615t^7 + 21\,420t^9 - 47\,250t^{11} + 44\,352t^{13} - 15\,015t^{15}) + \mathrm{O}(1/\nu^7)\bigg) \tag{A9}$$

$$s_l'e_l = \tfrac{1}{2}\bigg(1 + \frac{t^3}{2\nu} - \frac{1}{16\nu^3}(2t^3 - 27t^5 + 60t^7 - 35t^9) - \frac{1}{256\nu^5} \times (108t^5 - 3615t^7 + 21\,420t^9 - 47\,250t^{11} + 44\,352t^{13} - 15\,015t^{15}) + \mathrm{O}(1/\nu^7)\bigg). \tag{A10}$$

When deriving these expressions, the relations (A8) were taken into account.

It is useful to give also the expression for λ_l, defined in (2.6):

$$\lambda_l = \frac{t^3}{2\nu}\left(1 - \frac{1}{8\nu^2}(2 - 27t^2 + 60t^4 - 35t^6) - \frac{1}{128\nu^4}(108t^2 - 3615t^4 + 21\,420t^6 - 47\,250t^8 + 44\,352t^{10} - 15\,015t^{12}) + O(1/\nu^6)\right). \tag{A11}$$

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